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Nonlinear radiated and diffracted waves due to the motions of a submerged circular cylinder

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We study the high-harmonic surface wave associated with nonlinear diffraction and radiation of gravity waves by a near-surface circular cylinder. Based on nonlinear potential-flow theory, we show analytically, using a boundary-integral equation method, that: (*a*) for a circular motion of the cylinder, the leading-order outgoing radiated waves at any harmonic are generated only in one direction; (*b*) for a cylinder free to respond to regular incident waves, the total leading-order scattered waves at any harmonic are two orders smaller upstream of the body. These theoretical results are substantially confirmed by direct time-domain simulations of the problem.

1. Introduction

A significant aspect of nonlinear diffraction and radiation of surface waves by a body is the generation of high-harmonic waves. Although these high-harmonic waves generally have magnitudes that are higher order in wave steepness, they are important to ocean structures with high natural frequencies (and small damping at these frequencies), and to detection of (submerged) bodies by remote sensing (where ambient wave energy at these frequencies/wavelengths is small). Thus, the understanding and quantification of short-wavelength/high-harmonic waves associated with nonlinear wave–body interactions is of theoretical interest and practical engineering relevance.

A canonical problem in this context is the nonlinear interaction of waves with a submerged (two-dimensional) circular cylinder. A seminal work is that of Ogilvie (1963) who applied linear theory and employed the multipole expansions of Ursell (1950) to obtain three important results: (i) for a fixed cylinder, there is no wave reflection (see also Dean 1948); (ii) for a cylinder undergoing a circular motion, outgoing waves are generated in one direction only; and (iii) for an unrestrained (neutrally buoyant) body under incident waves, the total scattered waves vanish upstream.

For case (i) of wave reflection by a fixed circular cylinder, the linear result of Ogilvie (1963) has recently been extended to second-harmonic waves by Friis (1990), McIver & McIver (1990), and Wu (1991) with different approaches. Friis (1990) solved the second-order boundary-value problem for the velocity potential using a source-distribution method, while McIver & McIver (1990) and Wu (1991) obtained the second-order reflected wave using the first-order solution only. The most general result to date is that of Palm (1991), who showed that the leading-order component of any harmonic of the reflected wave vanishes. His analysis is based on the application

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of Green's theorem using the linear wave-source potential as the Green function. These theoretical predictions agree with the laboratory experiments of Chaplin (1984) and are supported by numerical results of Vada (1987) at second order and of Liu, Dommermuth & Yue (1992) up to third order.

The present work concerns the nonlinear (high-order/high-harmonic) extension of cases (ii) and (iii) of Ogilvie (1963), i.e. the cases when the cylinder is allowed to undergo captive (circular) or free motions. As in Palm (1991), we apply regular perturbation expansions and a boundary-integral equation method and solve the associated nonlinear boundary-value problems for the velocity potential up to an arbitrary high order in the wave steepness (or body motion). We obtain two principal results: (*a*) for a circular forced motion of the cylinder, the leading-order outgoing waves of any harmonic are generated only in one direction; and (*b*) for an untethered (neutrally buoyant) cylinder under incident waves, the leading-order outgoing scattered (combined diffracted and radiated) waves of any harmonic are two orders smaller upstream than downstream of the body. In short, the predictions of Ogilvie (1963) obtain to arbitrary high harmonic (to leading order). These theoretical results are confirmed by direct time-domain simulations of the problem using a high-order spectral method (Liu *et al.* 1992).

In § 2, we review the boundary-value problem for nonlinear wave interactions with a submerged circular cylinder and derive general formulas for the perturbation potentials in terms of free-surface and body-surface integrals. The analyses for the behaviour of the nonlinear outgoing waves are given in § 3 for the forced body motion problem, and in § 4 for the free response of the body under incident waves. In § 5, we perform numerical simulations to confirm our theoretical predictions. Conclusions and a discussion are given in § 6.

2. Mathematical formulation

We consider nonlinear diffraction and radiation of surface waves by a submerged circular cylinder in a fluid layer of infinite depth. The cylinder may undergo a forced circular oscillation or be free to respond to incident waves. The main focus is on obtaining the behaviour of the outgoing high-harmonic waves on either side of the body.

2.1. *The boundary-value problem*

We choose a global Cartesian coordinate system $\boldsymbol{x} = (x, y)$ with the x-axis in the quiescent free surface, x positive in the direction of incident wave propagation, y positive upward, and with the origin $(0, 0)$ directly above the (mean position of the) cylinder centre. We also place a local cylindrical coordinate system (r, θ) at the centre of the cylinder, which is at a (mean) depth h below the mean free surface. Thus, $r^2 = x^2 + (y + h)^2$, and θ is measured counter-clockwise from positive x.

We assume that the flow is irrotational, and that the fluid itself is homogeneous, incompressible, and inviscid. The flow can then be described by a velocity potential $\Phi(\boldsymbol{x}, t)$ which satisfies the Laplace equation ($\nabla^2\Phi = 0$) within the fluid, and vanishes at large depth ($\Phi \to 0$ as $y \to -\infty$). On the free surface, $y = \eta(x, t)$, the nonlinear boundary condition can be written in the form

$$\Phi_{tt} + g\Phi_z + 2\nabla\Phi \cdot \nabla\Phi_t + \tfrac{1}{2}\nabla\Phi \cdot \nabla(\nabla\Phi \cdot \nabla\Phi) = 0 \quad \text{on} \quad y = \eta(x, t), \tag{2.1}$$

where g is the gravitational acceleration. On the body $S_B(t)$, the condition that the fluid does not penetrate the body gives

$$\frac{\partial \Phi}{\partial n} = \boldsymbol{n} \cdot \boldsymbol{X}_t \quad \text{on} \quad S_B(t), \tag{2.2}$$

where $\boldsymbol{n} = (n_x, n_y)$ is the unit normal into the body and $\boldsymbol{X} = (X(t), Y(t))$ denotes the motion of the body. The boundary-value problem for Φ is complete with the imposition of an appropriate radiation condition as $|x| \to \infty$, in general, a physical requirement that the wave disturbances due to the body must propagate away from the body.

The free-surface elevation follows directly from the dynamic boundary condition on the free surface and is given by

$$\eta = -\frac{1}{g}(\Phi_t + \tfrac{1}{2}\nabla\Phi \cdot \nabla\Phi)|_{y=\eta}. \tag{2.3}$$

The pressure in the fluid is determined from Bernoulli's equation, and the force on the body can be obtained by integration of the pressure over the body surface. The motion of the body is governed by Newton's second law.

2.2. *Perturbation expansions and time-harmonic decompositions*

We assume that the steepness of surface waves is small and that the amplitude of the body motion compared to the body dimension is also small. For convenience, we employ the same small parameter, $\epsilon \equiv KA$ (or $\equiv K|\boldsymbol{X}|$) $\ll 1$, to measure the wave motion and body oscillation, where K and A are respectively the (fundamental) wavenumber and wave amplitude. Referring to ϵ, we expand the velocity potential Φ, the surface elevation η, and the body motion $\boldsymbol{X}$, in perturbation series:

$$\Phi = \Phi^{(1)} + \Phi^{(2)} + \ldots, \quad \eta = \eta^{(1)} + \eta^{(2)} + \ldots, \quad \text{and} \quad \boldsymbol{X} = \boldsymbol{X}^{(1)} + \boldsymbol{X}^{(2)} + \ldots, \tag{2.4}$$

where $(\,)^{(m)}$ denotes a quantity of $O(\epsilon^m)$.

We focus on the case of regular incident waves and time-periodic motions of the body and surface waves. We can then separate out the time dependences of the perturbation potentials, $\Phi^{(m)}$, $m = 1, 2, \ldots$, in terms of the fundamental frequency ω:

$$\Phi^{(1)}(\boldsymbol{x}, t) = \mathrm{Re}\{\phi_1^{(1)}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega t}\}, \tag{2.5}$$

$$\Phi^{(2)}(\boldsymbol{x}, t) = \mathrm{Re}\{\phi_0^{(2)}(\boldsymbol{x}) + \phi_2^{(2)}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}2\omega t}\}, \tag{2.6}$$

$$\Phi^{(m)}(\boldsymbol{x}, t) = \mathrm{Re}\{\phi_1^{(m)}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega t} + \phi_3^{(m)}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}3\omega t} + \cdots + \phi_m^{(m)}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}m\omega t}\}, \quad m > 2,\ m \text{ odd}, \tag{2.7}$$

$$\Phi^{(m)}(\boldsymbol{x}, t) = \mathrm{Re}\{\phi_0^{(m)}(\boldsymbol{x}) + \phi_2^{(m)}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}2\omega t} + \cdots + \phi_m^{(m)}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}m\omega t}\}, \quad m > 2,\ m \text{ even}. \tag{2.8}$$

In the above, $\phi_n^{(m)}$ denotes the (complex) amplitude of the mth-order, nth-harmonic potential, and clearly, the leading-order part of the mth harmonic solution is of mth order in magnitude, except that for the zeroth harmonic which is of second order. The same decomposition also applies to the surface elevation η and the body motion $\boldsymbol{X}$. Our interest is the leading-order behaviour of the solution at different harmonics, m, so the notation for the leading-order amplitude of the mth harmonic waves ($\phi_m^{(m)}$, say, for $m = 1, 2, \ldots$) will be simplified in what follows by omitting the subscript m (i.e. $\phi^{(m)}$).

We now derive the free-surface boundary condition for the perturbation potential $\phi^{(m)}, m = 1, 2, \ldots$. We introduce the perturbation expansions and harmonic decompositions for Φ and η into (2.1) and (2.3), and expand the free-surface condition (2.1) in a Taylor series about $y = 0$. Upon eliminating η, and collecting terms at each order and harmonic, we obtain a sequence of boundary conditions, applied on the mean free surface, for $\phi^{(m)}, m = 1, 2, \ldots$:

$$\phi_y^{(m)} - K_m\phi^{(m)} = f^{(m)}(x) \quad \text{on} \quad y = 0, \tag{2.9}$$

where $K_m \equiv m^2K = m^2\omega^2/g$. Here the free-surface forcing $f^{(m)}$ depends only upon lower-order potentials: $\phi^{(\ell)}$, $\ell < m$. For the first three orders, for example, $f^{(m)}$ is given by

$$f^{(1)} = 0, \tag{2.10}$$

$$\begin{aligned} f^{(2)} &\equiv \frac{\mathrm{i}\omega}{2g}\{\phi^{(1)}[\phi_{yy}^{(1)} - K\phi_y^{(1)}] - 2[\nabla\phi^{(1)}]^2\} \\ &= -\frac{\mathrm{i}\omega}{2g}\{3K^2[\phi^{(1)}]^2 + \phi^{(1)}\phi_{xx}^{(1)} + 2[\phi_x^{(1)}]^2\}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} f^{(3)} \equiv &-\frac{1}{8g}\{24\mathrm{i}\omega\nabla\phi^{(1)}\cdot\nabla\phi^{(2)} - 8K\phi^{(1)}\nabla\phi^{(1)}\cdot\nabla\phi_y^{(1)} + \nabla\phi^{(1)}\cdot\nabla[\nabla\phi^{(1)}]^2\} \\ &+\frac{\mathrm{i}\omega}{2g}\{\phi^{(1)}[\phi_{yy}^{(2)} - 4K\phi_y^{(2)}] + 2\phi^{(2)}[\phi_{yy}^{(1)} - K\phi_y^{(1)}]\} \\ &+\frac{1}{8g}\{2K\phi^{(1)}\phi_y^{(1)} + [\nabla\phi^{(1)}]^2\}[\phi_{yy}^{(1)} - K\phi_y^{(1)}] \\ = &-\frac{K^2}{8g}\{[\phi^{(1)}]^2[3K^2\phi^{(1)} - 7\phi_{xx}^{(1)}] + 3[\phi_x^{(1)}]^2[\phi_{xx}^{(1)}/K^2 + 4\phi^{(1)}] + \phi^{(1)}[\phi_{xx}^{(1)}]^2\} \\ &-\frac{\mathrm{i}\omega}{g}\{\phi^{(1)}[21K^2\phi^{(2)} + 5Kf^{(2)} + \phi_{xx}^{(2)}/2] + \phi_{xx}^{(1)}\phi^{(2)} + 3\phi_x^{(1)}\phi_x^{(2)}\}, \end{aligned} \tag{2.12}$$

where all quantities are evaluated at y=0.

Similarly, for the body boundary condition, we expand (2.2) in a Taylor series about the mean position of the body $\bar{S}_B$. Upon introducing the perturbation expansions and harmonic decompositions for Φ and $\boldsymbol{X}$, and collecting terms at each order and harmonic, we obtain a sequence of Neumann boundary conditions, applied on the mean body position, for $\phi^{(m)}, m = 1, 2, \ldots$:

$$\frac{\partial}{\partial n}\phi^{(m)}(\boldsymbol{x}) = b^{(m)}(\boldsymbol{x}) \quad \text{on} \quad \bar{S}_B, \tag{2.13}$$

where the body forcing $b^{(m)}$ is given in terms of the body motion and lower-order potentials $\phi^{(\ell)}$, $\ell < m$. For the first three orders, $b^{(m)}$ is given by

$$b^{(1)} = \mathrm{i}\omega\boldsymbol{n}\cdot\boldsymbol{\chi}^{(1)}, \tag{2.14}$$

$$b^{(2)} = 2\mathrm{i}\omega\boldsymbol{n}\cdot\boldsymbol{\chi}^{(2)} - \boldsymbol{n}\cdot\nabla\left[\boldsymbol{\chi}^{(1)}\cdot\nabla\phi^{(1)}\right]/2, \tag{2.15}$$

$$b^{(3)} = 3\mathrm{i}\omega\boldsymbol{n}\cdot\boldsymbol{\chi}^{(3)} - \boldsymbol{n}\cdot\nabla\left[\boldsymbol{\chi}^{(1)}\cdot\nabla\phi^{(2)} + \boldsymbol{\chi}^{(2)}\cdot\nabla\phi^{(1)}\right]/2 - \boldsymbol{n}\cdot\nabla\left[\boldsymbol{\chi}^{(1)}\cdot\nabla\right]^2\phi^{(1)}/4, \tag{2.16}$$

where all quantities are evaluated on $\bar{S}_B$, and $\boldsymbol{\chi}^{(m)}$ denotes the amplitude of the mth-harmonic body motion, which is of mth order.

In addition to the boundary conditions on the mean positions of the free surface and the body, $\phi^{(m)}$, $m = 1, 2, \ldots$, also need to satisfy the Laplace equation in the fluid domain, the condition at infinite depth, and the radiation condition as $|x| \to \infty$. At each order, m, the boundary-value problems for each perturbation potential $\phi^{(m)}$ are linear, and can be solved successively starting from $m = 1$.

In terms of $\phi^{(m)}$, the perturbation surface elevation $\eta^{(m)}$ can be obtained by Taylor expansion of (2.3) about the mean free surface. The perturbation pressure $p^{(m)}(\boldsymbol{x})$ in the fluid can be obtained from the Bernoulli equation and is given by

$$p^{(1)}/\rho = -\mathrm{i}\omega\phi^{(1)}, \tag{2.17}$$

$$p^{(m)}/\rho = -\mathrm{i}m\omega\phi^{(m)} - \frac{1}{4}\sum_{\ell=1}^{m-1} \nabla\phi^{(\ell)} \cdot \nabla\phi^{(m-\ell)}, \quad m \geqslant 2, \tag{2.18}$$

where ρ is the fluid density. Note that in (2.17) and (2.18), only the leading-order terms for the mth harmonic pressure ($m = 1, \ldots$) are included.

2.3. *The boundary-integral equation for* $\phi^{(m)}$

We define a Green function, $G(\boldsymbol{x}, \boldsymbol{x}', K_m)$, which is harmonic everywhere in the fluid except at $\boldsymbol{x}'$ where it is source like. In addition, G satisfies the linearized free-surface condition ($G_y - K_m G = 0$ on $y = 0$), the radiation condition ($G_x \pm \mathrm{i}K_m G = 0$ as $x \to \pm\infty$), and vanishes at large depth ($G \to 0$ as $y \to -\infty$). Physically, G represents the velocity potential due to a point source with pulsating strength, at frequency $m\omega$.

The solution for G is classical (e.g. Wehausen & Laitone 1960):

$$G(\boldsymbol{x}, \boldsymbol{x}', K_m) = \log r/r^* + G'(\boldsymbol{x}, \boldsymbol{x}', K_m), \tag{2.19}$$

where $r = [(x - x')^2 + (y - y')^2]^{1/2}$, $r^* = [(x - x')^2 + (y + y')^2]^{1/2}$, and G' can be written as

$$G' = -\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}k(\bar{z}-z')}}{k - K_m}\,\mathrm{d}k - \int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}k(z-\bar{z}')}}{k - K_m}\,\mathrm{d}k. \tag{2.20}$$

Here $z = x + \mathrm{i}y$, $\bar{z}$ is the complex conjugate of z, and $\int$ indicates that the path of integration is to go above the pole in the complex k-plane. As $|x| \to \infty$, the asymptotic behaviour of G can be obtained by contour integration:

$$G \sim 2\mathrm{i}\pi \mathrm{e}^{-\mathrm{i}K_m(z-\bar{z}')}, \quad x \to +\infty, \tag{2.21}$$

$$G \sim 2\mathrm{i}\pi \mathrm{e}^{\mathrm{i}K_m(\bar{z}-z')}, \quad x \to -\infty. \tag{2.22}$$

Applying Green's second identity to the potential $\phi^{(m)}$ and the Green function G, and using their boundary conditions on the free surface and at infinite depth, we obtain an integral equation for $\phi^{(m)}$:

$$\pi\phi^{(m)}(\boldsymbol{x}) - \int_{\bar{S}_B} \phi^{(m)} \frac{\partial G}{\partial n'}\,\mathrm{d}s' = -\int_{\bar{S}_B} G \frac{\partial \phi^{(m)}}{\partial n'}\,\mathrm{d}s' - \int_{S_F} f^{(m)} G'\,\mathrm{d}s' + \int_{S_\pm} \left[\phi^{(m)} \frac{\partial G}{\partial n'} - G \frac{\partial \phi^{(m)}}{\partial n'}\right] \mathrm{d}s', \tag{2.23}$$

for $\boldsymbol{x} \in \bar{S}_B$. In (2.23), a Cauchy principal-value integral is indicated on the left-hand side, $\partial/\partial n'$ denotes normal derivative out of the fluid, S_F represents the mean free surface on which $\log r/r^* = 0$, and $S_\pm$ denotes the boundaries far upstream and downstream of the body.

The integration over $S_\pm$ in (2.23) depends on the far-field behaviour of G and $\phi^{(m)}$. These are known for the Green function G (given by (2.21) and (2.22)). The far-field behaviour of $\phi^{(m)}$ depends on that of the free-surface forcing $f^{(m)}(x)$, and, for an arbitrary order m, is in general not known. For the special cases considered in this paper, however, we are able to show that at any order m, $f^{(m)} \to 0$ as $|x| \to \infty$. As a result, $\phi^{(m)}$ represents free propagating waves only in the far field.

To specify the radiation condition for $\phi^{(m)}$, we decompose $\phi^{(m)}$ into the incident wave $\phi_I^{(m)}$ and the disturbance (scattered) wave due to the body $\phi_D^{(m)}$:

$$\phi^{(m)} = \phi_I^{(m)} + \phi_D^{(m)}, \quad m = 1, 2, \ldots. \tag{2.24}$$

These satisfy the respective radiation conditions

$$\left(\partial/\partial x + \mathrm{i}K_m\right)\phi_I^{(m)} = 0, \quad x \to \pm\infty, \tag{2.25}$$

$$\left(\partial/\partial x \pm \mathrm{i}K_m\right)\phi_D^{(m)} = 0, \quad x \to \pm\infty, \tag{2.26}$$

where, without loss of generality, the incident wave is assumed to propagate in the positive x-direction.

Applying (2.25) and (2.26), and using the asymptotic values of G, we can evaluate the integrals over $S_\pm$ and rewrite (2.23) as

$$\begin{aligned}\pi\phi^{(m)}(\boldsymbol{x}) - ⨍_{\bar{S}_B} \phi^{(m)} \frac{\partial G}{\partial n'}\, \mathrm{d}s' = &- \int_{\bar{S}_B} G \frac{\partial \phi^{(m)}}{\partial n'}\, \mathrm{d}s' \\ &- \int_{S_F} f^{(m)} G'\, \mathrm{d}s' + 2\pi\phi_I^{(m)}(\boldsymbol{x}), \quad \boldsymbol{x} \in \bar{S}_B.\end{aligned} \tag{2.27}$$

At successive orders m, the body forcing $\partial\phi^{(m)}/\partial n'$, the free-surface forcing $f^{(m)}$, and the incident wave potential $\phi_I^{(m)}$ can be considered known. Hence, (2.27) is a second-kind Fredholm integral equation for the unknown potential $\phi^{(m)}$ on the body.

The potential $\phi^{(m)}$ anywhere in the fluid can be obtained from $\phi^{(m)}$ on $\bar{S}_B$. Applying Green's second identity to $\phi^{(m)}$ and G, and after using the radiation conditions and the boundary conditions on the free surface and at infinite depth, we obtain

$$2\pi[\phi^{(m)}(\boldsymbol{x}) - \phi_I^{(m)}(\boldsymbol{x})] = \int_{\bar{S}_B} \left[\phi^{(m)} \frac{\partial G}{\partial n'} - G \frac{\partial \phi^{(m)}}{\partial n'}\right] \mathrm{d}s' - \int_{S_F} f^{(m)} G'\, \mathrm{d}s', \tag{2.28}$$

for any $\boldsymbol{x}$ in the fluid.

2.4. *Solution of the integral equation*

Since all quantities on the body must be periodic functions of the polar angle θ, the integral equation (2.27) can be solved using a Fourier decomposition in θ. We expand the potential $\phi^{(m)}$ and its normal derivative $\partial\phi^{(m)}/\partial n$ on the body in Fourier series:

$$\phi^{(m)}(\theta) = \sum_{\ell=0}^{\infty} \alpha_{m\ell}^{+} \mathrm{e}^{\mathrm{i}\ell\theta} + \sum_{\ell=1}^{\infty} \alpha_{m\ell}^{-} \mathrm{e}^{-\mathrm{i}\ell\theta}, \tag{2.29}$$

$$\frac{\partial}{\partial n}\phi^{(m)}(\theta) = \sum_{\ell=1}^{\infty} \beta_{m\ell}^{+} \mathrm{e}^{\mathrm{i}\ell\theta} + \sum_{\ell=1}^{\infty} \beta_{m\ell}^{-} \mathrm{e}^{-\mathrm{i}\ell\theta}, \tag{2.30}$$

where $\alpha_{m\ell}^{\pm}$ and $\beta_{m\ell}^{\pm}$ are complex constants. For $\partial\phi^{(m)}/\partial n$, the zeroth mode does not need to be included since the net volume flux out of the body surface is zero.

Similarly, the Green function G and its normal derivative $\partial G/\partial n'$ can be expanded in Fourier series in θ and θ' (see e.g. Palm 1991). Upon using the relation $z = a\mathrm{e}^{\mathrm{i}\theta} - \mathrm{i}h$

for $z \in \bar{S}_B$, we have

$$\begin{aligned}
\log(r/r^*) &\equiv \mathrm{Re}\{\log(z-z') - \log(z-\bar{z}')\} \\
&= \mathrm{Re}\{\log \mathrm{e}^{\mathrm{i}\theta}[1-\mathrm{e}^{\mathrm{i}(\theta'-\theta)}] - \log 2\mathrm{i}h[1-(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta'})/2\mathrm{i}h]\} \\
&= \log 2h - \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell}[\mathrm{e}^{\mathrm{i}\ell(\theta'-\theta)} + \mathrm{e}^{-\mathrm{i}\ell(\theta'-\theta)}] \\
&\quad -\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{\mathrm{i}^{\ell}}{\ell(2h)^{\ell}}[(\mathrm{e}^{-\mathrm{i}\theta'}-\mathrm{e}^{\mathrm{i}\theta})^{\ell} + (\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta'})^{\ell}], \qquad \theta \neq \theta'; \qquad (2.31)
\end{aligned}$$

$$\begin{aligned}
G'(\theta,\theta',K_m) &\equiv -⨍_0^{\infty}\frac{\mathrm{e}^{-2kh}\mathrm{e}^{\mathrm{i}k(\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta'})}}{k-K_m}\,\mathrm{d}k - ⨍_0^{\infty}\frac{\mathrm{e}^{-2kh}\mathrm{e}^{-\mathrm{i}k(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta'})}}{k-K_m}\,\mathrm{d}k \\
&= -\sum_{\ell=0}^{\infty}\frac{\mathrm{i}^{\ell}}{\ell!}⨍_0^{\infty}\frac{k^{\ell}\mathrm{e}^{-2kh}}{k-K_m}\,\mathrm{d}k[(\mathrm{e}^{-\mathrm{i}\theta'}-\mathrm{e}^{\mathrm{i}\theta})^{\ell} + (\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta'})^{\ell}]; \qquad (2.32)
\end{aligned}$$

and

$$\frac{\partial}{\partial n'}G(\theta,\theta',K_m) = -\tfrac{1}{2} + \sum_{\ell=0}^{\infty}H_{\ell}[\mathrm{e}^{-\mathrm{i}\theta'}(\mathrm{e}^{-\mathrm{i}\theta'}-\mathrm{e}^{\mathrm{i}\theta})^{\ell} - \mathrm{e}^{\mathrm{i}\theta'}(\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta'})^{\ell}], \qquad (2.33)$$

with the constant H_{ℓ} given by

$$H_{\ell} = \frac{1}{2}\left(\frac{\mathrm{i}}{2h}\right)^{\ell+1} + \frac{\mathrm{i}^{\ell+1}}{\ell!}⨍_0^{\infty}\frac{k^{\ell+1}\mathrm{e}^{-2kh}}{k-K_m}\,\mathrm{d}k. \qquad (2.34)$$

Using (2.29)–(2.33), the body-surface integrals in (2.27) can be evaluated:

$$-⨍_{\bar{S}_B}\phi^{(m)}\frac{\partial G}{\partial n'}\,\mathrm{d}s' = \sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty}P^{+}_{\ell\ell'}\alpha^{+}_{m\ell'}\mathrm{e}^{\mathrm{i}\ell\theta} + \sum_{\ell=1}^{\infty}\sum_{\ell'=1}^{\infty}P^{-}_{\ell\ell'}\alpha^{-}_{m\ell'}\mathrm{e}^{-\mathrm{i}\ell\theta}, \qquad (2.35)$$

$$-\int_{\bar{S}_B}G\frac{\partial\phi^{(m)}}{\partial n'}\,\mathrm{d}s' = \sum_{\ell=0}^{\infty}\sum_{\ell'=1}^{\infty}Q^{+}_{\ell\ell'}\beta^{+}_{m\ell'}\mathrm{e}^{\mathrm{i}\ell\theta} + \sum_{\ell=1}^{\infty}\sum_{\ell'=1}^{\infty}Q^{-}_{\ell\ell'}\beta^{-}_{m\ell'}\mathrm{e}^{-\mathrm{i}\ell\theta}, \qquad (2.36)$$

where $P^{\pm}_{\ell\ell'}$ and $Q^{\pm}_{\ell\ell'}$ are known complex constants given in terms of mode numbers ℓ and ℓ', wavenumber K_m, and mean body submergence h.

After substituting G' and changing the order of integration, the free-surface integral in (2.27) becomes

$$-\int_{S_F}f^{(m)}(x')G'(\boldsymbol{x},\boldsymbol{x}')\,\mathrm{d}s' = 2\pi⨍_0^{\infty}\left[\frac{\mathrm{e}^{-\mathrm{i}kz}\tilde{f}^{(m)}(k)}{k-K_m} + \frac{\mathrm{e}^{\mathrm{i}k\bar{z}}\tilde{f}^{(m)}(-k)}{k-K_m}\right]\mathrm{d}k, \qquad (2.37)$$

where $\tilde{f}^{(m)}$ is the Fourier transform of $f^{(m)}(x)$ and has the form

$$\tilde{f}^{(m)}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}f^{(m)}(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x. \qquad (2.38)$$

Expanding the exponential functions in (2.37) in power series, it follows that

$$-\int_{S_F}f^{(m)}(x')G'(\boldsymbol{x},\boldsymbol{x}')\,\mathrm{d}s' = \sum_{\ell=0}^{\infty}\gamma^{+}_{m\ell}\mathrm{e}^{\mathrm{i}\ell\theta} + \sum_{\ell=1}^{\infty}\gamma^{-}_{m\ell}\mathrm{e}^{-\mathrm{i}\ell\theta}, \qquad (2.39)$$

where the Fourier modal amplitudes $\gamma^{\pm}$ are given by

$$\gamma^{\pm}_{m\ell} = 2\pi\delta_\ell \frac{(\mp \mathrm{i})^\ell}{\ell\,!} \int_0^\infty \frac{k^\ell \mathrm{e}^{-kh}}{k-K_m} \tilde{f}^{(m)}(\pm k)\,\mathrm{d}k, \quad \ell = 0,1,\ldots,\infty, \tag{2.40}$$

with $\delta_0 = 2$ and $\delta_\ell = 1$ for $\ell \geqslant 1$.

For the incident waves, $\phi_I^{(m)}(\boldsymbol{x})$ can also be expanded in a Fourier series:

$$2\pi\phi_I^{(m)}(\theta) = \sum_{\ell=0}^{\infty} \sigma^+_{m\ell}\mathrm{e}^{\mathrm{i}\ell\theta} + \sum_{\ell=1}^{\infty} \sigma^-_{m\ell}\mathrm{e}^{-\mathrm{i}\ell\theta}, \tag{2.41}$$

where $\sigma^{\pm}$ are the associated modal amplitudes.

Substituting (2.29), (2.30), (2.35), (2.36), (2.39), and (2.41) into the integral equation (2.27), and collecting coefficients for each Fourier mode, we obtain two uncoupled infinite sets of algebraic equations for the unknown amplitudes $\alpha^{\pm}_{m\ell}$:

$$\pi\alpha^+_{m\ell} + \sum_{\ell'=0}^{\infty} P^+_{\ell\ell'}\alpha^+_{m\ell'} = \gamma^+_{m\ell} + \sigma^+_{m\ell} + \sum_{\ell'=1}^{\infty} Q^+_{\ell\ell'}\beta^+_{m\ell'}, \quad \ell = 0,1,\ldots,\infty, \tag{2.42}$$

and

$$\pi\alpha^-_{m\ell} + \sum_{\ell'=1}^{\infty} P^-_{\ell\ell'}\alpha^-_{m\ell'} = \gamma^-_{m\ell} + \sigma^-_{m\ell} + \sum_{\ell'=1}^{\infty} Q^-_{\ell\ell'}\beta^-_{m\ell'}, \quad \ell = 1,2,\ldots,\infty. \tag{2.43}$$

A notable feature of (2.42) and (2.43) is that α^+ (or α^-) depends only on the positive (or negative) Fourier modes of the forcing of the integral equation (2.27).

For $\boldsymbol{x}$ on S_F, the Green function $G = G'$. Upon substituting G' into (2.28), the body integrals can be evaluated by using (2.29) and (2.30) and then expanding the exponential functions ($\mathrm{e}^{-\mathrm{i}kz'}$ and $\mathrm{e}^{\mathrm{i}k\bar{z}'}$) into power series. Changing the order of integration for the free-surface integral, we obtain from (2.28) the velocity potential due to the body on the mean free surface:

$$\phi_D^{(m)}(\boldsymbol{x}) = \int_0^\infty \left[\frac{C_m^+(k)}{k-K_m}\mathrm{e}^{-\mathrm{i}kx} + \frac{C_m^-(k)}{k-K_m}\mathrm{e}^{\mathrm{i}kx} \right] \mathrm{d}k, \quad y = 0, \tag{2.44}$$

with $C_m^{\pm}$ given by

$$C_m^{\pm}(k) = \tilde{f}^{(m)}(\pm k) + \mathrm{e}^{-kh} \sum_{\ell=1}^{\infty} \frac{(\pm \mathrm{i}k)^\ell}{\ell\,!} (\ell\alpha^{\pm}_{m\ell} + \beta^{\pm}_{m\ell}). \tag{2.45}$$

Applying the asymptotic values of G at the far field and evaluating the body integrals in (2.28), we obtain the asymptotic solutions of the body disturbance potential:

$$\phi_D^{(m)}(\boldsymbol{x}) \sim -2\mathrm{i}\pi C_m^+(K_m)\mathrm{e}^{-\mathrm{i}K_m z}, \quad x \to +\infty, \tag{2.46}$$

$$\phi_D^{(m)}(\boldsymbol{x}) \sim -2\mathrm{i}\pi C_m^-(K_m)\mathrm{e}^{\mathrm{i}K_m \bar{z}}, \quad x \to -\infty. \tag{2.47}$$

3. Cylinder forced to oscillate in a circular orbit

We study the problem of wave radiation due to a forced circular motion of a submerged circular cylinder. From linearized analysis, it is known that outgoing (first-harmonic) waves are generated in one direction only (Ogilvie 1963). We shall generalize this result to arbitrary high-harmonic waves by including nonlinear boundary effects of the free surface and body.

Without loss of generality, the body is considered to perform a clockwise circular motion with frequency ω and amplitude R, i.e. $\boldsymbol{X}(t) = \mathrm{Re}\{\boldsymbol{\chi}^{(1)}\mathrm{e}^{\mathrm{i}\omega t}\}$ with $\boldsymbol{\chi}^{(1)} = (\mathrm{i}R, -R)$ and $\boldsymbol{\chi}^{(m)} = (0,0)$ for $m = 2,3,\ldots$. No incident wave is involved so $\phi_I^{(m)} = 0$ for any m. For this case, we shall show that outgoing waves of any harmonic propagate toward the right (positive x) direction only, i.e. $\phi^{(m)} \to 0$ as $x \to -\infty$ for any m.

3.1. *First-harmonic waves*

At first order, m=1, the free-surface forcing $f^{(1)} \equiv 0$ and thus $\gamma_{1\ell}^- = 0$, $\ell = 1,2,\ldots$ (cf. (2.40)). On the body, $\boldsymbol{n} = -(\cos\theta, \sin\theta)$. From (2.14), the body forcing $b^{(1)} = \mathrm{i}\omega\boldsymbol{n}\cdot\boldsymbol{\chi}^{(1)} = \omega R\mathrm{e}^{\mathrm{i}\theta}$, which leads to $\beta_{1\ell}^- = 0$, $\ell = 1,2,\ldots$ (cf. (2.30)). From (2.43) and (2.45), it directly follows that $\alpha_{1\ell}^- = 0$, $\ell = 1,2,\ldots$, and $C_1^- = 0$. Hence, we have from (2.47) that $\phi^{(1)} = 0$ as $x \to -\infty$.

The potential on the body then becomes (cf. (2.29))

$$\phi^{(1)}(\theta) = \sum_{\ell=0}^{\infty} \alpha_{1\ell}^+ \mathrm{e}^{\mathrm{i}\ell\theta}. \tag{3.1}$$

The potential on the mean free surface follows from (2.44):

$$\phi^{(1)}(x) = ⨍_0^{\infty} \frac{C_1^+(k)}{k-K}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k, \quad y = 0. \tag{3.2}$$

From (2.46), the asymptotic value of the potential far downstream is obtained to be

$$\phi^{(1)}(\boldsymbol{x}) \sim -2\mathrm{i}\pi C_1^+(K)\mathrm{e}^{-\mathrm{i}Kz}, \quad x \to +\infty. \tag{3.3}$$

3.2. *Second-harmonic waves*

At second order, $m = 2$, $f^{(2)}(x)$ vanishes far upstream owing to $\phi^{(1)} = 0$ as $x \to -\infty$, and the substitution of (3.3) into (2.11) leads to $f^{(2)}(x) = 0$ as $x \to +\infty$. Thus, $\phi^{(2)}$ contains only free waves in the far field and (2.26) is the proper form of the radiation condition. From (3.2), it is clear that the Fourier transform of $\phi^{(1)}(x)$, denoted by $\tilde{\phi}^{(1)}(k)$, vanishes for negative values of k. Since $f^{(2)}$ consists of products of $\phi^{(1)}$ and its x-derivatives (cf. (2.11)), we can evaluate $\tilde{f}^{(2)}$ through convolutions of $\tilde{\phi}^{(1)}$, $\tilde{\phi}_x^{(1)}$, and $\tilde{\phi}_{xx}^{(1)}$, and obtain that $\tilde{f}^{(2)}(k) = 0$ for $k < 0$. This gives $\gamma_{2\ell}^- = 0$, $\ell = 1,2,\ldots$ (cf. (2.40)).

With the relation $\partial/\partial n = -\partial/\partial r$, we rewrite the body forcing $b^{(2)}$ in (2.15) as

$$b^{(2)}(\boldsymbol{x}) = \frac{R}{2}\frac{\partial}{\partial r}\left(\mathrm{i}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\phi^{(1)}(\boldsymbol{x}), \quad \boldsymbol{x} \in \bar{S}_B. \tag{3.4}$$

Using the coordinate transform relations $r_x = \cos\theta$, $r_y = \sin\theta$, $\theta_x = -\sin\theta/r$, and $\theta_y = \cos\theta/r$, we convert the x- and y-derivatives in (3.4) into r- and θ-derivatives to obtain

$$b^{(2)}(\theta) = \mathrm{e}^{\mathrm{i}\theta}\frac{R}{2}\left(\mathrm{i}\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\right)\phi^{(1)}(r,\theta), \quad r = a. \tag{3.5}$$

From Laplace's equation, we have

$$\frac{\partial^2\phi^{(1)}}{\partial r^2} = \left[-\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi^{(1)}. \tag{3.6}$$

Substituting (3.6) into (3.5), it becomes clear that $b^{(2)}$ is specified in terms of θ-derivatives of $\phi^{(1)}$ and $\phi_r^{(1)}$ only. From the first-order solution, it is known that $\phi^{(1)}(\theta)$

and $\phi_r^{(1)}(\theta)$ on the body contain only positive Fourier modes, i.e. $\mathrm{e}^{\mathrm{i}\ell\theta}$, $\ell > 0$. As a result, $b^{(2)}(\theta)$ must not contain any negative Fourier modes. Thus $\beta_{2\ell}^-$ vanishes for any ℓ.

Upon obtaining that $\tilde{f}^{(2)}(k<0)$, $\gamma_{2\ell}^-$, $\beta_{2\ell}^- = 0$, it follows that $\alpha_{2\ell}^- = 0$, $\ell = 1, 2, \ldots$ (cf. (2.43)), and then $C_2^- = 0$ (cf. (2.45)). From (2.47), we find $\phi^{(2)} = 0$ as $x \to -\infty$. The second-order potential on the body thus takes the form

$$\phi^{(2)}(\theta) = \sum_{\ell=0}^{\infty} \alpha_{2\ell}^+ \mathrm{e}^{\mathrm{i}\ell\theta}. \tag{3.7}$$

The potential on the mean free surface and its asymptotic value far downstream become

$$\phi^{(2)}(x) = \fint_0^\infty \frac{C_2^+(k)}{k - K_2} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}k, \quad y = 0, \tag{3.8}$$

and

$$\phi^{(2)}(\boldsymbol{x}) \sim -2\mathrm{i}\pi C_2^+(K_2)\mathrm{e}^{-\mathrm{i}K_2 z}, \quad x \to +\infty. \tag{3.9}$$

3.3. *Third-harmonic waves*

From (2.12), we see that the third-order free-surface forcing $f^{(3)}$ consists of three groups of terms: (i) $[\phi^{(1)}\nabla\phi^{(1)}\cdot\nabla\phi_y^{(1)}]$ and $[\nabla\phi^{(1)}\cdot\nabla\phi^{(2)}]$; (ii) $[\nabla\phi^{(1)}\cdot\nabla(\nabla\phi^{(1)})^2]$; and (iii) $[\phi_{yy}^{(1)} - K\phi_y^{(1)}]$ and $[\phi_{yy}^{(2)} - K_2\phi_y^{(2)}]$. Far downstream, $\phi^{(1)}, \phi^{(2)} = 0$ and thus $f^{(3)}$ vanishes as $x \to -\infty$. Far upstream, both $\phi^{(1)}$ and $\phi^{(2)}$ are outgoing waves (in infinite depth) and thus $(\nabla\phi^{(1)})^2$, $\nabla\phi^{(1)}\cdot\nabla\phi^{(2)}$, $[\phi_{yy}^{(1)} - K\phi_y^{(1)}]$, and $[\phi_{yy}^{(2)} - K_2\phi_y^{(2)}]$ all vanish as $x \to +\infty$ (cf. (3.3) and (3.9)). Whence, $f^{(3)}$ also vanishes as $x \to +\infty$. It then follows that $\phi^{(3)}$ represents free waves only in the far field.

Using the free-surface condition (2.9) and Laplace's equation, the y-derivatives of $\phi^{(1)}$ and $\phi^{(2)}$ on $y = 0$ can be evaluated in terms of their x-derivatives. From (2.12), $f^{(3)}(x)$ essentially consists of products of $\phi^{(1)}$, $\phi^{(2)}$, $f^{(2)}$, and their x-derivatives only. Since $\tilde{\phi}^{(1)}(k)$, $\tilde{\phi}^{(2)}(k)$, and $\tilde{f}^{(2)}(k)$ vanish for negative values of k, it follows that $\tilde{f}^{(3)}(k) = 0$ for $k < 0$.

For the body forcing, we first rewrite $b^{(3)}$ in (2.16) as

$$b^{(3)}(\boldsymbol{x}) = \frac{R}{2}\frac{\partial}{\partial r}\left(\mathrm{i}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\phi^{(2)}(\boldsymbol{x}) + \frac{R^2}{4}\frac{\partial}{\partial r}\left(\mathrm{i}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2\phi^{(1)}(\boldsymbol{x}), \quad \boldsymbol{x} \in \bar{S}_B. \tag{3.10}$$

Upon changing x-, y-derivatives into r-, θ-derivatives, we obtain

$$\begin{aligned} b^{(3)}(\theta) &= \frac{R^2}{4}\frac{\partial}{\partial r}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\frac{\partial}{\partial r} - \frac{\mathrm{e}^{\mathrm{i}\theta}}{r}\frac{\partial}{\partial\theta}\right)^2\phi^{(1)}(\theta) + \mathrm{e}^{\mathrm{i}\theta}\frac{R}{2}\frac{\partial}{\partial r}\left(\mathrm{i}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial\theta}\right)\phi^{(2)}(\theta) \\ &= \mathrm{e}^{2\mathrm{i}\theta}\frac{R^2}{4}\frac{\partial}{\partial r}\left(-\frac{\partial^2}{\partial r^2} + \frac{\mathrm{i}}{r^2}\frac{\partial}{\partial\theta} - \frac{2\mathrm{i}}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi^{(1)}(\theta) \\ &\quad + \mathrm{e}^{\mathrm{i}\theta}\frac{R}{2}\frac{\partial}{\partial r}\left(\mathrm{i}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial\theta}\right)\phi^{(2)}(\theta), \quad r = a. \end{aligned} \tag{3.11}$$

Similarly to (3.6), the following relations can be derived from Laplace's equation:

$$\frac{\partial^3}{\partial r^2\partial\theta} = -\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial^3}{\partial\theta^3}, \quad \text{and} \quad \frac{\partial^3}{\partial r^3} = \frac{2}{r^2}\frac{\partial}{\partial r} + \frac{3}{r^3}\frac{\partial^2}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^3}{\partial r\partial\theta^2}. \tag{3.12}$$

Using (3.6) and (3.12), it follows that $b^{(3)}$ in (3.11) consists of terms proportional to θ-derivatives of $\phi^{(1)}$, $\phi_r^{(1)}$, $\phi^{(2)}$, and $\phi_r^{(2)}$ only. Since $\phi^{(1)}$, $\phi^{(2)}$ as well as their first normal derivatives on the body contain positive Fourier modes only, $b^{(3)}$ does not possess terms proportional to $e^{i\ell\theta}$, $\ell < 0$. Thus, $\beta_{3\ell}^-$ vanishes for $\ell = 1, 2, \ldots$.

From (2.43) and (2.45), we obtain that $\alpha_{3\ell}^- = 0$ for any ℓ and $C_3^- = 0$, which gives $\phi^{(3)} = 0$ as $x \to -\infty$ (cf. (2.47)). The third-order potentials on the body, on the mean free surface, and far downstream are respectively given by

$$\phi^{(3)}(\theta) = \sum_{\ell=0}^{\infty} \alpha_{3\ell}^+ e^{i\ell\theta}, \tag{3.13}$$

$$\phi^{(3)}(x) = \int_0^{\infty} \frac{C_3^+(k)}{k - K_3} e^{-ikx}\, dk, \quad y = 0, \tag{3.14}$$

$$\phi^{(3)}(\boldsymbol{x}) \sim -2i\pi C_3^+(K_3) e^{-iK_3 z}, \quad x \to +\infty. \tag{3.15}$$

3.4. *mth-harmonic waves*, $m \geqslant 4$.

At arbitrary mth order, the free-surface forcing $f^{(m)}$ consists of products of potentials $\phi^{(n)}$, $n \leqslant m-1$, and their derivatives. As in the case of $m = 3$, $f^{(m)}$ can be divided into three groups of terms, where groups (i) and (ii) consists of terms proportional to $\nabla\phi^{(n)} \cdot \nabla\phi^{(n)}$ and its y-derivatives, while group (iii) contains terms of $\phi_y^{(n)} - K_n\phi^{(n)}$ and its y-derivatives. If $\phi^{(n)}$, $n \leqslant m-1$ is zero far downstream, and possesses a solution proportional to $\sim e^{iK_n z}$ far upstream, $\nabla\phi^{(n)} \cdot \nabla\phi^{(n)}$, $\phi_y^{(n)} - K_n\phi^{(n)}$, and their y-derivatives all vanish. Hence $f^{(m)} = 0$ as $|x| \to \infty$ and $\phi^{(m)}$ contains only free waves in the far field.

Using Laplace's equation and the free-surface condition (2.9), the y-derivatives of $\phi^{(n)}$ on $y = 0$ can be evaluated in terms of $\phi^{(n)}$, $f^{(n)}$ and their x-derivatives. In follows that the free-surface forcing terms $f^{(m)}$ are also given in terms of products of $\phi^{(n)}$, $f^{(n)}$, $n = 1, \ldots, m-1$, and their x-derivatives. Since $\tilde{\phi}^{(n)}(k)$ and $\tilde{f}^{(n)}(k)$, $n = 1, \ldots, m-1$, vanish for negative values of k, $\tilde{f}^{(m)}(k) = 0$ for $k < 0$ as a result of convolutions of $\tilde{\phi}^{(n)}(k)$ and $\tilde{f}^{(n)}(k)$. From (2.40), we thus obtain $\gamma_{m\ell}^- = 0$, $\ell = 1, 2, \ldots$.

On the body, the forcing $b^{(m)}$ can be written in a general form:

$$\begin{aligned} b^{(m)}(\theta) &= -\sum_{q=1}^{m-1} \left(\frac{1}{q!2^q}\right) \boldsymbol{n} \cdot \nabla[\boldsymbol{\chi}_1^{(1)} \cdot \nabla]^q \phi^{(m-q)}(\boldsymbol{x}) \\ &= \sum_{q=1}^{m-1} \frac{R^q}{q!2^q} \frac{\partial}{\partial r} \left(ie^{i\theta} \frac{\partial}{\partial r} - \frac{e^{i\theta}}{r} \frac{\partial}{\partial \theta} \right)^q \phi^{(m-q)}(\theta), \quad r = a. \end{aligned} \tag{3.16}$$

Using the Laplace equation, high-order r-derivatives of $\phi^{(n)}$ in (3.16) can be evaluated in terms of θ-derivatives of $\phi^{(n)}$ and $\phi_r^{(n)}$. Hence, $b^{(m)}$ in (3.16) can be written in terms proportional to θ-derivatives of $\phi^{(n)}$ and $\phi_r^{(n)}$ only. For $n < m$, $\phi^{(n)}$ and its normal derivatives on the body are shown to be given by positive Fourier modes only. Thus, $b^{(m)}$ must also contain only positive Fourier modes, i.e. $\beta_{m\ell}^- = 0$, $\ell = 1, 2, \ldots$.

From (2.43), (2.45), and (2.47), it follows that $\alpha_{m\ell}^- = 0$ for any ℓ, $C_m^- = 0$ and thus $\phi^{(m)} = 0$ as $x \to -\infty$. Finally, the potential on the body is given by

$$\phi^{(m)}(\theta) = \sum_{\ell=0}^{\infty} \alpha_{m\ell}^+ e^{i\ell\theta}. \tag{3.17}$$

The potential on the mean free surface and its behaviour far downstream are given by

$$\phi^{(m)}(x) = \int_0^{\infty} \frac{C_m^+(k)}{k-K_m} \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k, \quad y=0, \tag{3.18}$$

$$\phi^{(m)}(\boldsymbol{x}) \sim -2\mathrm{i}\pi C_m^+(K_m)\mathrm{e}^{-\mathrm{i}K_m z}, \quad x \to +\infty. \tag{3.19}$$

The corresponding surface elevation $\eta^{(m)}$ is given in terms of products of $\phi^{(n)}$, $n \leqslant m$ and their derivatives. Since all these quantities are zero as $x \to -\infty$, the leading-order outgoing waves of any harmonic vanish far upstream of the body.

4. Cylinder free to respond to an incident wave

We consider the interaction of an incident wave with a submerged neutrally buoyant circular cylinder which is free to move in response to the wave exciting forces. According to linear theory (Ogilvie 1963), the body undergoes a circular motion and there are no reflected (first-harmonic) waves. In this section, we include nonlinearities of the free-surface and body boundary conditions to generalize this result to any high harmonic m.

Note that we do not include the time-averaged drift motion of the cylinder in the analysis. From the free-surface and body boundary conditions in the presence of a steady forward motion (e.g. Newman 1978), it is clear that the effect of the drift motion (which is second-order) upon the harmonic solution, $\phi^{(m)}$, $m=1,\ldots$, is $(m+2)$th order. Thus, the drift motion does not affect the leading-order harmonic results we seek.

For incident waves, we employ a Stokes wave for which $\phi_I^{(m)} = 0$ for any value of m except $\phi_I^{(1)}$ which is given by

$$\phi_I^{(1)} = \frac{\mathrm{i}gA}{\omega}\mathrm{e}^{-\mathrm{i}Kz}, \tag{4.1}$$

where A is the incident wave amplitude. Upon expanding the exponential in (4.1) in a power series, and using the relation $z = a\mathrm{e}^{\mathrm{i}\theta} - \mathrm{i}h$, we obtain that $\sigma_{1\ell}^- = 0$, $\ell = 1, 2, \ldots$, in (2.41).

4.1. *First-harmonic waves*

At first order, $m=1$, $\gamma_{1\ell}^{\pm} = 0$ for any ℓ, as a result of $f^{(1)} = 0$. For the incident waves, $\sigma_{1\ell}^- = 0$, but $\sigma_{1\ell}^+ \neq 0$, for $\ell = 1, 2, \ldots$. In response to the incident waves, the body undergoes a periodic oscillation $\boldsymbol{X}^{(1)}(t) = \mathrm{Re}\{\boldsymbol{\chi}^{(1)}\mathrm{e}^{\mathrm{i}\omega t}\}$. Note that the motion amplitude $\boldsymbol{\chi}^{(1)} = (\xi^{(1)}, \zeta^{(1)})$ is unknown and to be determined from the equation of motion. In terms of $\xi^{(1)}$ and $\zeta^{(1)}$, the forcing on the body can be expressed as

$$b^{(1)}(\boldsymbol{x}) = -\frac{\mathrm{i}}{2}\omega\{[\xi^{(1)} - \mathrm{i}\zeta^{(1)}]\mathrm{e}^{\mathrm{i}\theta} + [\xi^{(1)} + \mathrm{i}\zeta^{(1)}]\mathrm{e}^{-\mathrm{i}\theta}\}, \quad \boldsymbol{x} \in \bar{S}_B. \tag{4.2}$$

From (4.2), we obtain that $\beta_{1\ell}^{\pm}$ vanishes for any ℓ except $\beta_{11}^{\pm} = -\mathrm{i}\omega[\xi^{(1)} \mp \mathrm{i}\zeta^{(1)}]/2$.

From (2.42) and (2.43), it is clear that $\alpha_{1\ell}^{\pm} \neq 0$ for any ℓ. For convenience, we write these in the form

$$\alpha_{1\ell}^+ = [\xi^{(1)} - \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{1\ell}^+ + \check{\alpha}_{1\ell}^+, \quad \alpha_{1\ell}^- = [\xi^{(1)} + \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{1\ell}^-, \quad \ell = 1, 2, \ldots, \tag{4.3}$$

where $\hat{\alpha}_{1\ell}^{\pm}$ is due to the body motion ($\beta_{11}^{\pm}$) of unit amplitude, and $\check{\alpha}_{1\ell}^+$ is due to the incident waves ($\sigma_{1\ell}^+$). The potential on the body can now be expressed as

$$\phi^{(1)}(\theta) = \sum_{\ell=0}^{\infty}\{\check{\alpha}_{1\ell}^+ + [\xi^{(1)} - \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{1\ell}^+\}\mathrm{e}^{\mathrm{i}\ell\theta} + [\xi^{(1)} + \mathrm{i}\zeta^{(1)}]\sum_{\ell=1}^{\infty}\hat{\alpha}_{1\ell}^-\mathrm{e}^{-\mathrm{i}\ell\theta}. \tag{4.4}$$

Integrating the pressure (2.17) over the body and using (4.4), we obtain the force on the body:

$$F_x^{(1)} = a\int_0^{2\pi} p_1^{(1)} n_x \,\mathrm{d}\theta = \pi \mathrm{i}\omega\rho a\{\check{\alpha}_{11}^{+} + [\xi^{(1)} - \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{11}^{+} + [\xi^{(1)} + \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{11}^{-}\}, \tag{4.5}$$

$$F_y^{(1)} = a\int_0^{2\pi} p_1^{(1)} n_y \,\mathrm{d}\theta = -\pi\omega\rho a\{\check{\alpha}_{11}^{+} + [\xi^{(1)} - \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{11}^{+} - [\xi^{(1)} + \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{11}^{-}\}. \tag{4.6}$$

Upon applying Newton's second law, it follows that

$$\mathrm{i}\omega a\xi^{(1)} = \check{\alpha}_{11}^{+} + [\xi^{(1)} - \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{11}^{+} + [\xi^{(1)} + \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{11}^{-}, \tag{4.7}$$

$$\omega a\zeta^{(1)} = \check{\alpha}_{11}^{+} + [\xi^{(1)} - \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{11}^{+} - [\xi^{(1)} + \mathrm{i}\zeta^{(1)}]\hat{\alpha}_{11}^{-}. \tag{4.8}$$

Solving (4.7) and (4.8) for $\xi^{(1)}$ and $\zeta^{(1)}$, we obtain

$$\xi^{(1)} = -\mathrm{i}\zeta^{(1)} = \check{\alpha}_{11}^{+}/(\mathrm{i}\omega a - 2\hat{\alpha}_{11}^{+}). \tag{4.9}$$

Thus $\xi^{(1)}$ and $\zeta^{(1)}$ have the same magnitude, but differ in phase by $\pi/2$, i.e. the body motion is a clockwise circular one (relative to an incident waves propagating from left to right).

With $\xi^{(1)}$ and $\zeta^{(1)}$ given by (4.9), it follows that $\beta_{1\ell}^{-}, \alpha_{1\ell}^{-} = 0$, $\ell = 1, 2, \ldots$ (cf. (4.2) and (4.3)). Hence, $C_1^{-} = 0$ from (2.45) and $\phi_D^{(1)} = 0$ as $x \to -\infty$ from (2.47). The total potential on the body is

$$\phi^{(1)}(\theta) = \sum_{\ell=0}^{\infty} \frac{\mathrm{i}\omega a\check{\alpha}_{1\ell}^{+}}{\mathrm{i}\omega a - 2\hat{\alpha}_{11}^{+}} \mathrm{e}^{\mathrm{i}\ell\theta}. \tag{4.10}$$

The potential on the mean free surface is found to be (cf. (2.44) and (4.1))

$$\phi^{(1)}(x) = \frac{\mathrm{i}gA}{\omega}\mathrm{e}^{-\mathrm{i}Kx} + ⨍_0^{\infty} \frac{C_1^{+}(k)}{k-K}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k, \quad y = 0, \tag{4.11}$$

and its asymptotic values in far field are given by (cf. (2.46), (2.47) and (4.1))

$$\phi^{(1)}(\boldsymbol{x}) \sim \left[\mathrm{i}gA/\omega - 2\mathrm{i}\pi C_1^{+}(K)\right]\mathrm{e}^{-\mathrm{i}Kz}, \quad x \to +\infty, \tag{4.12}$$

$$\phi^{(1)}(\boldsymbol{x}) \sim \frac{\mathrm{i}gA}{\omega}\mathrm{e}^{-\mathrm{i}Kz}, \quad x \to -\infty. \tag{4.13}$$

4.2. *Second-harmonic waves*

As in the forced motion problem, the substitution of (4.12) and (4.13) into (2.11) leads to $f^{(2)}(x) = 0$ as $|x| \to \infty$. Since $\tilde{\phi}^{(1)}(k) = 0$ for $k < 0$ (cf. (4.11)), $\tilde{f}^{(2)}(k)$ vanishes for negative values of k (cf. (2.11)). This leads to $\gamma_{2\ell}^{-} = 0$, $\ell = 1, 2, \ldots$.

On the body, we separate the body forcing into two parts, $b^{(2)}(\boldsymbol{x}) = b_1^{(2)}(\boldsymbol{x}) + b_2^{(2)}(\boldsymbol{x})$, where $b_1^{(2)}$, due to the second-harmonic motion of the body, is given by

$$b_1^{(2)}(\boldsymbol{x}) = 2\mathrm{i}\omega\boldsymbol{n}\cdot\boldsymbol{\chi}^{(2)} = -\mathrm{i}\omega\{[\xi^{(2)} - \mathrm{i}\zeta^{(2)}]\mathrm{e}^{\mathrm{i}\theta} + [\xi^{(2)} + \mathrm{i}\zeta^{(2)}]\mathrm{e}^{-\mathrm{i}\theta}\}, \quad \boldsymbol{x} \in \bar{S}_B\ ; \tag{4.14}$$

and $b_2^{(2)}$, associated with the first-order solution, has the form

$$b_2^{(2)}(\boldsymbol{x}) = \frac{|\xi^{(1)}|}{2}\frac{\partial}{\partial r}\left(\mathrm{i}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\phi^{(1)}(\boldsymbol{x}), \quad \boldsymbol{x} \in \bar{S}_B. \tag{4.15}$$

Correspondingly, the Fourier components of $b^{(2)}(\boldsymbol{x})$ can be expressed as

$$\beta_{2\ell}^{\pm} = \hat{\beta}_{2\ell}^{\pm} + \check{\beta}_{2\ell}^{\pm}, \quad \ell = 1, 2, \ldots, \tag{4.16}$$

where $\hat{\beta}$ denotes the effect of $b_1^{(2)}$, while $\check{\beta}$ is associated with $b_2^{(2)}$. From (4.14), it follows that $\hat{\beta}_{2\ell}^{\pm}$ vanishes for any ℓ except $\check{\beta}_{21}^{\pm} = -\mathrm{i}\omega[\xi^{(2)} \mp \mathrm{i}\zeta^{(2)}]$. For the same reason as in the forced body motion case, $b_2^{(2)}$ contains terms proportional to $\mathrm{e}^{\mathrm{i}\ell\theta}$, $\ell > 0$, only, and hence $\check{\beta}_{2\ell}^{-} = 0$, for $\ell > 0$.

As in the first-order problem, we separate $\alpha_{2\ell}^{\pm}$ into two parts:

$$\alpha_{2\ell}^{+} = [\xi^{(2)} - \mathrm{i}\zeta^{(2)}]\hat{\alpha}_{2\ell}^{+} + \check{\alpha}_{2\ell}^{+}, \quad \alpha_{2\ell}^{-} = [\xi^{(2)} + \mathrm{i}\zeta^{(2)}]\hat{\alpha}_{2\ell}^{-}, \quad \ell = 1, 2, \ldots, \tag{4.17}$$

where $\hat{\alpha}_{2\ell}^{\pm}$ is due to second-harmonic body motion ($\beta_{21}^{\pm}$) of unit amplitude, and $\check{\alpha}_{2\ell}^{+}$ is due to free-surface forcing ($\gamma_{2\ell}^{+}$) and body forcing ($\check{\beta}_{2\ell}^{+}$). Upon integrating the second-order pressure (2.18) over the body, we obtain the force on the body:

$$F_x^{(2)} = 2\pi\mathrm{i}\omega\rho a\{\check{\alpha}_{21}^{+} + [\xi^{(2)} - \mathrm{i}\zeta^{(2)}]\hat{\alpha}_{21}^{+} + [\xi^{(2)} + \mathrm{i}\zeta^{(2)}]\hat{\alpha}_{21}^{-}\} + \pi\mathrm{i}\rho a\mathscr{F}^{(2)}, \tag{4.18}$$

$$F_y^{(2)} = -2\pi\omega\rho a\{\check{\alpha}_{21}^{+} + [\xi^{(2)} - \mathrm{i}\zeta^{(2)}]\hat{\alpha}_{21}^{+} - [\xi^{(2)} + \mathrm{i}\zeta^{(2)}]\hat{\alpha}_{21}^{-}\} - \pi\rho a\mathscr{F}^{(2)}, \tag{4.19}$$

where $\mathscr{F}^{(2)}$ is given by

$$\mathscr{F}^{(2)} = -\frac{\mathrm{i}}{4\pi}\int_0^{2\pi}\left[\nabla\phi_1^{(1)}\right]^2 \mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}\theta. \tag{4.20}$$

Applying the equation of motion, we obtain

$$\xi^{(2)} = -\mathrm{i}\zeta^{(2)} = \frac{\check{\alpha}_{21}^{+} + \mathscr{F}^{(2)}/2\omega}{2\mathrm{i}\omega a - 2\hat{\alpha}_{21}^{+}}. \tag{4.21}$$

As for the first-order body response, $\xi^{(2)}$ and $\zeta^{(2)}$ have equal magnitude but a phase difference of $\pi/2$, and so the second-order response is also a clockwise circular motion.

Because of (4.21), $\beta_{2\ell}^{-}, \alpha_{2\ell}^{-} = 0$, $\ell = 1, 2, \ldots$ (cf. (4.14) and (4.17)). Thus, $C_2^{-} = 0$ from (2.45) and $\phi^{(2)} = 0$ as $x \to -\infty$ from (2.47). The total potential on the body is then

$$\phi^{(2)}(\theta) = \sum_{\ell=0}^{\infty} \frac{\mathrm{i}\omega a\check{\alpha}_{2\ell}^{+} + \hat{\alpha}_{2\ell}^{+}\mathscr{F}^{(2)}/2\omega}{\mathrm{i}\omega a - \hat{\alpha}_{21}^{+}}\mathrm{e}^{\mathrm{i}\ell\theta}. \tag{4.22}$$

The total potential on the mean free surface and far downstream are given by

$$\phi^{(2)}(x) = \int_0^{\infty} \frac{C_2^{+}(k)}{k - K_2}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k, \quad y = 0; \tag{4.23}$$

$$\phi^{(2)}(\boldsymbol{x}) \sim -2\mathrm{i}\pi C_2^{+}(K_2)\mathrm{e}^{-\mathrm{i}K_2 z}, \quad x \to +\infty. \tag{4.24}$$

4.3. *mth-harmonic waves, $m \geqslant 3$*

Following the same argument as for the forced-body oscillation case, the free-surface forcing $f^{(m)}(x)$ vanishes as $|x| \to \infty$, and its Fourier components $\tilde{f}^{(m)}(k) = 0$ for $k < 0$. We thus obtain from (2.40) that $\gamma_{m\ell}^{-} = 0$ for any ℓ.

For the body forcing, as in the case of $m = 2$, we separate $b^{(m)}(\boldsymbol{x})$ into two parts: $b^{(m)} = b_1^{(m)} + b_2^{(m)}$. The first part, $b_1^{(m)}$, is due to mth-harmonic oscillation of the body and has the form

$$b_1^{(m)}(\boldsymbol{x}) = \mathrm{i}m\omega\boldsymbol{n}\cdot\boldsymbol{\chi}^{(m)} = -\frac{\mathrm{i}}{2}m\omega\{[\xi^{(m)} - \mathrm{i}\zeta^{(m)}]\mathrm{e}^{\mathrm{i}\theta} + [\xi^{(m)} + \mathrm{i}\zeta^{(m)}]\mathrm{e}^{-\mathrm{i}\theta}\}, \quad \boldsymbol{x} \in \bar{S}_B. \tag{4.25}$$

The second part, $b_2^{(m)}$, is associated with the interactions of lower-order solutions, and is given by

$$b_2^{(m)}(\boldsymbol{x}) = \sum_{q=1}^{m-1} \frac{1}{q!2^q} \boldsymbol{n} \cdot \nabla [\boldsymbol{\chi}^{(1)} \cdot \nabla]^q \phi^{(m-q)}(\boldsymbol{x})$$
$$+ \sum_{q=1}^{m-2} \frac{1}{(q-1)!2^q} \boldsymbol{n} \cdot \nabla [\boldsymbol{\chi}^{(1)} \cdot \nabla]^{(q-1)} [\boldsymbol{\chi}^{(2)} \cdot \nabla] \phi^{(m-q-1)}(\boldsymbol{x}) + \ldots, \tag{4.26}$$

for $\boldsymbol{x} \in \bar{S}_B$, where in the above, $\cdots$ represents all mth-order interactions between the body motion $\boldsymbol{\chi}^{(1)}, \ldots, \boldsymbol{\chi}^{(m-1)}$ and the potential $\phi^{(1)}, \ldots, \phi^{(m-1)}$. Correspondingly, the Fourier components of $b^{(m)}(\boldsymbol{x})$ can be expressed as

$$\beta_{m\ell}^{\pm} = \hat{\beta}_{m\ell}^{\pm} + \check{\beta}_{m\ell}^{\pm}, \quad \ell = 1, 2, \ldots, \tag{4.27}$$

where $\hat{\beta}$ denotes the effect of $b_1^{(m)}$, while $\check{\beta}$ is due to $b_2^{(m)}$. Equation (4.25) indicates that $\hat{\beta}_{m\ell}^{\pm}$ vanishes for any ℓ except $\hat{\beta}_{m1}^{\pm} = -\mathrm{i}m\omega[\xi^{(m)} \mp \mathrm{i}\zeta^{(m)}]/2$.

Since the body undergoes clockwise circular motions at lower orders, it follows that

$$\boldsymbol{\chi}^{(n)} \cdot \nabla = |\xi^{(n)}| \left(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} \frac{\partial}{\partial r} - \frac{\mathrm{e}^{\mathrm{i}\theta}}{r} \frac{\partial}{\partial \theta} \right), \quad n = 1, 2, \ldots. \tag{4.28}$$

Substituting (4.28) into (4.26), it can be seen that $b_2^{(m)}(\boldsymbol{x})$ consists of products of $\mathrm{e}^{\mathrm{i}\ell\theta}$, $\ell > 0$, and the derivatives of $\phi^{(n)}$, $n \leqslant m-1$. Using Laplace's equation, higher-order r-derivatives of $\phi^{(n)}$ can be converted into θ-derivatives of $\phi^{(n)}$ and $\phi_r^{(n)}$. Since $\phi^{(n)}$ and $\phi_r^{(n)}$ on the body do not contain negative Fourier modes, $b_2^{(m)}(\boldsymbol{x})$ consists of terms proportional to $\mathrm{e}^{\mathrm{i}\ell\theta}$, $\ell > 0$ only. It follows that $\check{\beta}_{m\ell}^{-} = 0$ for $\ell = 1, 2, \ldots$.

Based on (2.42) and (2.43), we can write $\alpha_{m\ell}^{\pm}$ as

$$\alpha_{m\ell}^{+} = [\xi^{(m)} - \mathrm{i}\zeta^{(m)}]\hat{\alpha}_{m\ell}^{+} + \check{\alpha}_{m\ell}^{+}, \quad \alpha_{m\ell}^{-} = [\xi^{(m)} + \mathrm{i}\zeta^{(m)}]\hat{\alpha}_{m\ell}^{-}, \quad \ell = 1, 2, \ldots. \tag{4.29}$$

Here $\hat{\alpha}_{m\ell}^{\pm}$ results from the mth-harmonic body motion ($\beta_{m1}^{\pm}$), while $\check{\alpha}_{m\ell}^{+}$ is associated with the free-surface forcing ($\gamma_{m\ell}^{+}$) and the body forcing ($\check{\beta}_{m\ell}^{+}$).

Upon integrating the pressure (2.18) over the body, we obtain the force on the body:

$$F_x^{(m)} = m\pi\mathrm{i}\omega\rho a\{\check{\alpha}_{m1}^{+} + [\xi^{(m)} - \mathrm{i}\zeta^{(m)}]\hat{\alpha}_{m1}^{+} + [\xi^{(m)} + \mathrm{i}\zeta^{(m)}]\hat{\alpha}_{m1}^{-}\} + \pi\mathrm{i}\rho a\mathscr{F}^{(m)}, \tag{4.30}$$

$$F_y^{(m)} = -m\pi\omega\rho a\{\check{\alpha}_{m1}^{+} + [\xi^{(m)} - \mathrm{i}\zeta^{(m)}]\hat{\alpha}_{m1}^{+} - [\xi^{(m)} + \mathrm{i}\zeta^{(m)}]\hat{\alpha}_{m1}^{-}\} - \pi\rho a\mathscr{F}^{(m)}, \tag{4.31}$$

where $\mathscr{F}^{(m)}$ is given by

$$\mathscr{F}^{(m)} = -\frac{\mathrm{i}}{4\pi} \sum_{q=1}^{m-1} \int_0^{2\pi} [\nabla\phi^{(q)} \cdot \nabla\phi^{(m-q)}] \mathrm{e}^{-\mathrm{i}\theta} \, \mathrm{d}\theta. \tag{4.32}$$

After applying Newton's equation of motion, we have

$$\xi^{(m)} = -\mathrm{i}\zeta^{(m)} = \frac{\check{\alpha}_{m1}^{+} + \mathscr{F}^{(m)}/m\omega}{\mathrm{i}m\omega a - 2\hat{\alpha}_{m1}^{+}}. \tag{4.33}$$

As with the lower-order solutions for the body responses, $\xi^{(m)}$ and $\zeta^{(m)}$ have the same magnitude but differ in phase by $\pi/2$, and thus describe a clockwise circular motion of the body.

Using (4.33), we have from (4.25) and (4.29) that $\beta^-_{m\ell}$, $\alpha^-_{m\ell} = 0$, $\ell = 1, 2, \ldots$. Consequently, $C^-_m = 0$ (cf. (2.45)) and hence $\phi^{(m)} = 0$ as $x \to -\infty$ (cf. (2.47)). The total potential on the body has the form:

$$\phi^{(m)}(\theta) = \sum_{\ell=0}^{\infty} \frac{\mathrm{i}m\omega a\check{\alpha}^+_{m\ell} + 2\hat{\alpha}^+_{m\ell}\mathscr{F}^{(m)}/m\omega}{\mathrm{i}m\omega a - 2\hat{\alpha}^+_{m1}} \mathrm{e}^{\mathrm{i}\ell\theta} \; ; \tag{4.34}$$

and the potential on the mean free surface and far downstream are obtained to be

$$\phi^{(m)}(x) = \int_0^{\infty} \frac{C^+_m(k)}{k - K_m} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}k, \quad y = 0, \tag{4.35}$$

$$\phi^{(m)}(\boldsymbol{x}) \sim -2\mathrm{i}\pi C^+_m(K_m)\mathrm{e}^{-\mathrm{i}K_m z}, \quad x \to +\infty. \tag{4.36}$$

The corresponding surface elevation $\eta^{(m)}$ is given in terms of products of $\phi^{(n)}$, $n \leqslant m$, and its derivatives. Since all these quantities are zero as $x \to -\infty$ except $\phi^{(1)}$ which is equal to $\phi^{(1)}_I$, $\eta^{(m)}$ has the value given by the incident wave, and thus the leading-order reflected waves of any harmonic vanish far upstream of the body.

5. Numerical confirmation

As illustration, and to obtain numerical support of our theoretical predictions, we perform time-domain simulations of the nonlinear interactions between surface waves and a submerged circular cylinder to obtain high-order high-harmonic results. For completeness, we consider not only the cases of forced circular motion (§ 3), and free motion in an incident wave (§ 4), but obtain also numerical confirmation for the case of a fixed cylinder considered by Palm (1991).

The numerical method we use is an extension of the high-order spectral (HOS) method of Liu *et al.* (1992) to include body motions. In the present method, the body boundary condition is satisfied at its instantaneous position. If the body is not fixed or captive, its motion at any time is obtained by integrating the equations of motion resulting from Newton's second law with the forces on the body provided by pressure integration using

$$p(\theta, t)/\rho = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} + (\dot{\boldsymbol{X}} - \tfrac{1}{2}\nabla\Phi)\cdot\nabla\Phi, \tag{5.1}$$

where $\mathrm{d}/\mathrm{d}t$ represents the substantial derivative and $\dot{\boldsymbol{X}}$ the body velocity.

In the HOS method, the free surface and body surface are represented respectively by dipole and source distributions given in terms of Fourier spectral series (with N_F, N_B modes respectively). The HOS method follows the evolution of the free-surface waves using a pseudo-spectral treatment of the nonlinear free-surface conditions and accounts for nonlinear interactions among the N_F wave modes and N_B source modes up to an arbitrary order M in wave steepness. For moderately steep waves, the method exhibits exponential convergence with respect to the order M and the number of spectral modes N_F, N_B. In addition, with the use of fast-Fourier transforms, the computational effort is only linearly proportional to N_F (typically $N_F \gg N_B$), so that in practice very large values of N_F can be used to obtain extremely high accuracy results. Details of the HOS implementation and performance can be found in Liu *et al.* (1992) and Liu (1994) and are omitted here.

5.1. *Decomposition of the nonlinear wave field*

In the nonlinear time-domain simulation, we obtain a limit-cycle (steady-state) solution of the nonlinear total wave field $\eta(x,t)$ around the body after some time. To compare with theory, it is necessary to extract the free-wave information at each harmonic frequency from $\eta(x,t)$. The total wave field $\eta(x,t)$ contains two components: the incident waves and the body disturbance, $\eta(x,t) = \eta_I(x,t) + \eta_D(x,t)$. The incident wave is known and it is only necessary to consider the harmonic decomposition of the disturbance wave field:

$$\eta_D(x,t) = \sum_{m=0}^{\infty} \eta_D^{(m)}(x)\mathrm{e}^{\mathrm{i}m\omega t} + \text{c.c.}, \tag{5.2}$$

where c.c. represents the complex conjugate of the preceding term. At each harmonic, $\eta_D^{(m)}(x)$ contains free propagating, locked, and evanescent waves:

$$\eta_D^{(m)}(x) = \eta_F^{(m)}(x) + \eta_L^{(m)}(x) + \eta_E^{(m)}(x), \quad m = 1, 2, \ldots, \tag{5.3}$$

where η_F, η_L, and η_E represent free, locked, and evanescent wave components, respectively. A free wave satisfies the linear dispersion relation, and has a spatial dependence of the form

$$\eta_F^{(m)}(x) = A_F^{(m)}\mathrm{e}^{\mathrm{i}K_m x}, \quad m = 1, 2, \ldots, \tag{5.4}$$

where $A_F^{(m)}$ denotes the complex amplitude of the mth-harmonic free wave.

A locked wave does not satisfy the dispersion relation and has multiple wavenumbers. Physically, a locked wave is the response to forcing on the free surface due to nonlinear combinations of free waves at that frequency. Given free-wave components with wavenumbers K_m and frequencies $m\omega$, there are locked waves $\eta_L^{(m)}$ at frequency $m\omega$ resulting from combination of free-wave components of frequency $\ell_i\omega$, satisfying $\sum_i \ell_i = m$. For a given m, there may be J_m such frequency combinations, with the associated wavenumbers for the locked waves given by $K_{mj} = \sum_i K_i$. Consequently, $\eta_L^{(m)}$ has the general form

$$\eta_L^{(m)}(x) = \sum_{j=1}^{J_m} A_{Lj}^{(m)}\mathrm{e}^{\mathrm{i}K_{mj}x}, \quad m = 2, 3, \ldots, \tag{5.5}$$

where $A_{Lj}^{(m)}$ is the amplitude of the jth-component of the mth-harmonic locked waves. For $m = 1$, there is no locked wave, $\eta_L^{(1)}(x) = 0$.

At each harmonic, evanescent waves have the same frequency but are non-propagating and localized near the body with attenuation rates with distance governed by local geometry/body submergence. The far-field behaviour of such evanescent disturbances can be obtained by approximating the body as a dipole located at the (mean) body centre, so that asymptotically

$$\eta_E^{(m)} \sim A_e^{(m)}/|K_m x|^2, \quad |K_m x| \gg 1, \tag{5.6}$$

where $A_e^{(m)}$ denotes the amplitude of mth-harmonic evanescent waves.

With the (far-field) spatial dependences of $\eta_F^{(m)}$, $\eta_L^{(m)}$ and $\eta_E^{(m)}$ at each harmonic explicitly given by (5.4), (5.5) and (5.6), the unknown harmonic amplitudes can be obtained simply, say, by collocating (5.3) at an appropriate set of discrete points on the (mean) free surface where the values of the surface elevation can be obtained from simulation.

ϵ	$\lvert\eta^{(1)}_{F_-}\rvert/\lvert\eta^{(1)}_{F_+}\rvert$	$\lvert\eta^{(2)}_{F_-}\rvert/\lvert\eta^{(2)}_{F_+}\rvert$	$\lvert\eta^{(3)}_{F_-}\rvert/\lvert\eta^{(3)}_{F_+}\rvert$	ϵ^2
0.025	0.00012	0.00016	0.00013	0.00063
0.050	0.00058	0.00028	0.00016	0.00250
0.075	0.00290	0.00170	0.00180	0.00560
0.100	0.01000	0.01100	0.01700	0.01000

TABLE 1. Ratios of the amplitudes of the reflected and transmitted waves of the first three harmonics by a fixed submerged circular cylinder, radius $Ka = 0.25$, submergence $h/a = 2$, for different incident wave steepness, $\epsilon \equiv KA$. The results are obtained using the HOS method with $M = 4$, $N_F = 1024$, $N_B = 64$.

R/a	$\lvert\eta^{(1)}_{F_-}\rvert/\lvert\eta^{(1)}_{F_+}\rvert$	$\lvert\eta^{(2)}_{F_-}\rvert/\lvert\eta^{(2)}_{F_+}\rvert$	$\lvert\eta^{(3)}_{F_-}\rvert/\lvert\eta^{(3)}_{F_+}\rvert$	$\epsilon^2 \equiv (K\eta_{max})^2$
0.05	0.00008	0.00004	0.00002	0.00007
0.10	0.00036	0.00053	0.00046	0.00026
0.15	0.00095	0.00073	0.00060	0.00068
0.20	0.00190	0.00160	0.00140	0.00145

TABLE 2. Ratios of the amplitudes of the left and right radiated waves of the first three harmonics due to the forced clockwise circular motion of a submerged circular, radius $Ka = 0.25$, submergence $h/a = 2$, for different motion amplitudes, R/a. The results are obtained using the HOS method with $M = 4$, $N_F = 1024$, $N_B = 64$.

5.2. *Numerical results*

We choose a cylinder radius of $Ka = 0.25$ and a mean submergence of $h/a = 2$. The computational domain has a length equal to 32 fundamental wavelengths with the body located in the middle of the domain. All numerical results are obtained with order $M = 4$, and number of free-surface and body modes $N_F = 1024$, $N_B = 64$. With these computational parameters, the results for surface wave elevations up to third harmonic are established to be accurate to at least the fourth decimal place.

Case (1). *Wave diffraction by a fixed cylinder.*

Numerical results for the ratios of the amplitudes of reflected ($\eta^{(m)}_{F_-}$) and transmitted ($\eta^{(m)}_{F_+}$) waves for $m = 1, 2, 3$, are presented in table 1 for a range of incident wave steepness ϵ (a column for ϵ^2 is also given for convenience). It is seen that in all cases the reflected waves are two orders smaller than the transmitted waves. This is in support of the theoretical prediction of Palm (1991) concerning the vanishing of the leading-order reflected waves of any harmonic for this problem.

Case (2). *Wave radiation by a cylinder in forced circular motion.*

For the cylinder in a clockwise circular orbit, table 2 gives the ratios of the amplitudes of the radiated waves far upstream ($\eta^{(m)}_{F_-}$) and far downstream ($\eta^{(m)}_{F_+}$) for a range of motion amplitudes R/a. These results show that for clockwise circular motion, the leading-order outgoing waves at each harmonic are two orders (in wave steepness) smaller to the left (η_{F_-}) than to the right (η_{F_+}) at least up to $m = 3$. This confirms the analysis in § 3.

Case (3). *Wave scattering by a cylinder free to respond.*

In this case, we first confirm that the harmonic motions are indeed clockwise circular (for an incident wave from left to right) as obtained in § 4. Of main interest here are the amplitudes of the total scattered waves far upstream and downstream of the cylinder. These results are given in table 3 in terms of the ratios of these

ϵ	$\lvert\eta_{F_-}^{(1)}\rvert/\lvert\eta_{F_+}^{(1)}\rvert$	$\lvert\eta_{F_-}^{(2)}\rvert/\lvert\eta_{F_+}^{(2)}\rvert$	$\lvert\eta_{F_-}^{(3)}\rvert/\lvert\eta_{F_+}^{(3)}\rvert$	ϵ^2
0.021	0.00031	0.00043	0.00091	0.00044
0.116	0.01368	0.02000	0.02000	0.01346
0.188	0.02852	0.09830	0.09010	0.03534

TABLE 3. Ratios of the amplitudes of the upstream and downstream scattered waves of the first three harmonics by a neutrally-buoyant submerged circular cylinder, radius $Ka = 0.25$, submergence $h/a = 2$, for different incident wave steepness, $\epsilon \equiv KA$. The results are obtained using the HOS method with $M = 4$, $N_F = 1024$, $N_B = 64$.

amplitudes for the first three harmonics for a range of incident wave steepness ϵ. The numerical results confirm the theoretical prediction in §4 regarding vanishing of the leading-order scattered wave far upstream. As pointed out in §4, drift motion does not affect the leading-order results. This is also confirmed numerically (by allowing or disallowing the cylinder to drift). The results here are for the cylinder not allowed to drift.

6. Conclusions and discussion

We study the outgoing waves resulting from nonlinear radiation and diffraction by a submerged horizontal circular cylinder undergoing a forced circular motion or free motion in respond to incident waves. We show that: (*a*) for the cylinder undergoing forced circular motions, the leading-order outgoing waves of any harmonic are generated in one direction only (toward the right for a clockwise motion); and (*b*) for a neutrally buoyant cylinder under incident waves, the leading-order scattered wave of any harmonic vanishes upstream of the body. These results are generalizations to arbitrary high order/harmonic of the classical linear results of Ogilvie (1963) for a submerged cylinder in motion. We confirm these theoretical predictions up to third order by nonlinear time-domain simulations.

The analysis in §4 for the reflection of monochromatic incident waves by an untethered cylinder can be extended, in a straightforward manner, to bichromatic and multichromatic incident waves. For two incident waves with frequencies ω_1 and ω_2, an analogous result can be obtained that the leading-order waves of frequency $m\omega_1 + n\omega_2$ (m, n arbitrary positive integers) are not reflected and in response, the body sustains a circular motion of frequency $m\omega_1 + n\omega_2$ at $(m + n)$th order. For multiple incident waves, the reflected waves of any sum frequency are two orders smaller in magnitude than the corresponding transmitted waves.

Owing to the use of perturbation expansions, the present analysis of the leading-order solution for any harmonic is strictly valid only for (moderately) small surface wave and body motion amplitudes. For large-amplitude waves or body motions, higher-order corrections to the leading-order solution at any harmonic, which are two orders smaller, may become appreciable and need to be included. In this situation, the present predictions of vanishing upstream waves may no longer hold (see e.g. Wu 1993; Liu *et al.* 1992).

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